GRAPH FACTORIZATION AND ITS APPLICATION

Jayanta Kr. Choudhury*

Anupam Dutta**

Bichitra Kalita***

Abstract:

In this paper, different types of factorization of graphs of the complete graphs K_{6m-2}, K_{6m+2} and K_{6m} for m≥1 have been studied. An algorithm for the solution of TSP has been developed. Some theoretical investigations related to 3-factors, 2-factors and 1-factors have been discussed. Finally, some experimental results have been cited.

Key Words: Traveling salesman problem (TSP), Factors of graph, Hamiltonian Circuit.

* Swadeshi Academy Junior College, Guwahati-781005

** Patidarrang College. Muktapur, Loch, Kamrup, Assam.

*** Department of Computer Application, Assam Engg. College, Guwahati-781013.

1.0. Introduction:

The traveling salesman problem has long been of great interest topic related to Hamiltonian graphs. It is turn out to be an important topic for the graph theory and other branches of science too. It is known that the traveling salesman problem is an NP-complete problem and one has to study the Hamiltonian circuit related to minimum distance, minimum time, minimum cost etc. from the weighted graph. The traveling salesman problem is still an unsolved problem only because of the fact that no efficient method is available. Many heuristic methods are available for solution of the traveling salesman problem in certain cases and they are found to be not up-to-date level to determine the shortest route (actual route) of the traveler. Kalita [2] has studied the traveling salesman problem and he forwarded some heuristic methods for solution of the traveling salesman problem in some specialized cases. Recently Kalita [2] has developed a theory which gave the number of non-isomorphic Hamiltonian sub graphs of the form H(2m+3,6m+3) for m≥2 obtained from the complete graph K_{2m+3} for m≥2. Besides,he has developed an algorithm for TSP. Further, the relationship of these types of graphs with the metal atom of cluster compound in chemistry has been focused by Kalita [2]. Moreover, a heuristic method of traveling salesman problem under different situations for the complete graph K_{2m+3} for m≥2 has also been forwarded to determine the least cost route of a traveler for complete weighted graph when the weights are non-repeated. Recently, it has been found [7] that the application of regular planar sub graphs of the complete graph K_{2m+2} for m≥2 plays an important role in some special situation of traveling salesman problem. Some theoretical results have also been discussed, and finally they have developed an algorithm corresponding to the planar sub-graph of the complete graph K_{2m+2} for m≥2 to find the shortest path for TSP. Very recently, a heuristic method [9] for traveling salesman problem under different cases of the complete weighted graph K_{2m+3} for m≥2 has also been forwarded when the weights are repeated. An algorithm of traveling salesman problem has been discussed in [9] under different cases i.e., when 2m+2 consecutive greatest weights for m≥2 are incident with a vertex and 2 ≤ number of equal weights ≤ 2m are incident with another vertex. Two theorems have also been forwarded by Choudhury and Kalita in [9]. In addition to this, an algorithm has also been developed.

The paper is organized as follows: The section 1.0 explains some related works on Traveling Salesman Problems. In section 1.1, the notation and terminology have been included with some definitions. Theoretical investigation has been included in section 2.0.An algorithm has been included in section 2.1. The experimental results have been cited in section 2.2 and the conclusion in section 2.3.

1.1. Notation and Terminology:

The notation and terminology are considered from standard references [1-10]. For the graph G(n, e), 'n' denotes the number of vertices, 'e' the number of edges. The number of 3-factor, 2-factor and 1-factor are respectively denoted by S, T and L. We consider the complete graph as weighted graph when we apply them in case of traveling salesman problems.

Definitions:

(a) **1-factor**: 1-factor is a sub-graph of a graph G where each of the vertices are of degree one and union of these sub-graphs forms the original graph. Suppose K_4 is a complete graph then there are three 1-factors which are shown in figure-A

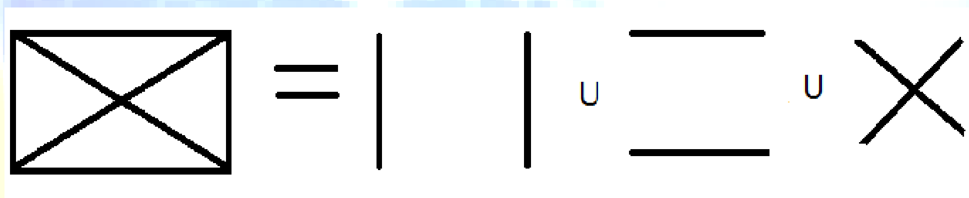

Figure-A

(b) **2-Factor:** 2-Factor is a sub-graph of a graph G where each of the vertices are of degree two and their union forms the original graph. Similarly the definition of 3-factor, 4-factor,….., n-factor can be defined [10].

2.0. Theoretical investigations:

The following theorem has special attention for factorization of graphs

Theorem1. For the complete graph K_{6m-2} for m≥1, S=2m-1, that is, the number of three factors is 2m-1.

Proof: We are going to prove the theorem with the help of method of mathematical induction. Here, the complete graph K_{6m-2} for $m \geq 1$ has $n=6m-2$ number of vertices and $e=18m^2-15m+3$ number of edges.

Let $m=1$. Then we have the complete graph K_4 in which the number of vertices $n=4$ and the number of edges $e=6$.Obviously, it has only one 3- factor i.e., it has only one three regular sub-graph, that is the graph itself. That is, $2m-1=1=S$ for $m=1$, which is shown in figure-1. So, the theorem is true for $m=1$.

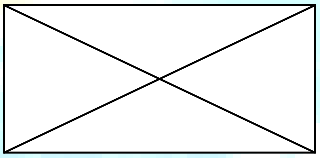

Figure-1

Again, if we suppose $m=2$, then we have the complete graph K_{10}, in which the number of vertices $n=10$ and the number of edges $e=45$. Clearly, it can be shown that $S=2m-1=4-1=3$. That is, the number of three regular sub-graphs is 3 and each three regular sub-graph contains fifteen edges. This is true for the complete graph K_{10}.

Let us consider that the theorem is true for $m=k$. Then the form of the complete graph is K_{6k-2}, in which number of vertices $n=6k-2$, number of edges $e=18k^2-15k+3$ and number of three regular sub-graph $S=2k-1$. Now, if we put $m=k+1$, then the complete graph is $K_{6(k+1)-2}$ where number of vertices $n=6(k+1)-2$ and number of edges $e=18(k+1)^2-15(k+1)+3$. Similarly, $S=2(k+1)-1=2k+1$.

As we have considered the values of $m \geq 1$ for all cases, hence $m \geq 1 \Rightarrow k+1 \geq 1 \Rightarrow k \geq 0$, which shows that $S=2k+1$ for $k \geq 0$

Hence the theorem is true for all $m \geq 1$.

Theorem2. The complete graph K_{6m+2} for $m \geq 1$ has $S=2m$ and $L=1$.

Proof: We are going to prove the theorem with the help of method of mathematical induction. Here, the complete graph K_{6m+2} for $m \geq 1$ has $n=6m+2$ number of vertices and $e=18m^2+9m+1$ number of edges.

Let m=1. Then we have the complete graph K_8 in which the number of vertices n=8 and the number of edges e =28.Obviously it has one 1-factor graph and two 3- factor graphs and union of these three graphs give K_8.That is, 2m=2=S for m=1 and L=1 which is shown in Figure-2. So, the theorem is true for m=1.

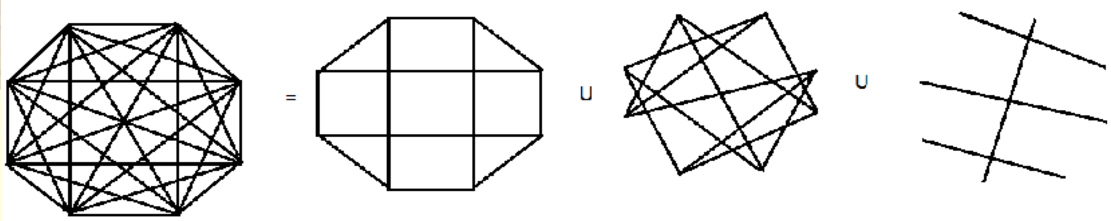

Figure-2

Again if we suppose m=2, then we have the complete graph K_{14}, in which the number of vertices n =14 and the number of edges e=91. Clearly, it can be shown that 2m=4=S. That is, the number of three regular sub-graph is 4 and one 1-factor graph is one i.e. L=1, i.e. each three factor graph has e=21 and 1-factor graph has e=7. This shows that the theorem is true for m=2

Let us consider that the theorem is true for m=k. Then the form of the complete graph is K_{6k+2}, in which number of vertices n=6k+2, number of edges $e=18k^2+9k+1$ and number of three regular sub-graph S=2k. Now, if we put m=k+1, then the complete graph is $K_{6(k+1)+2}$ where number of vertices n=6(k+1)+2=6k+8 and number of edges $e=18(k+1)^2+9(k+1)+1=18k^2+45k+28$. Similarly, S=2(k+1)=2k+2 and L=1.

Now, m≥1=>k+1≥1=>k≥0, which shows that S=2k+2 for k≥0.

Hence the theorem is true for all m≥1.

Theorem3. The complete graph K_{6m} for m≥1 has S=2m-1 and T=1.

Proof: We are going to prove the theorem with the help of method of mathematical induction. Here, the complete graph K_{6m} for m≥1 and it has number of vertices n=6m, and number of edges $e=18m^2-3m$.

Let m=1. Then we have the complete graph K_6 in which the number of vertices n=6 and the number of edges e =15.Obviously it has one 2-factor graph and one 3- factor graph and union of these two graphs we will have K_6 That is, 2m-1=1=S for m=1 and T=1 which is shown in Figure-3.

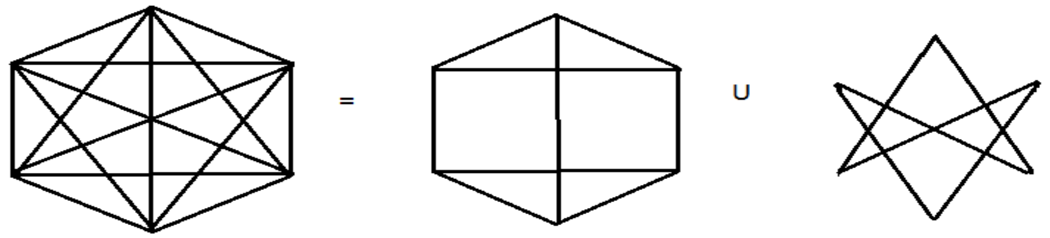

FIG-3

Again if we suppose m=2, then we have the complete graph K_{12}, in which the number of vertices n =12 and the number of edges e=66. Clearly, it can be shown that 2m-1=3=S. That is, the number of three regular sub-graph is 3 and one 2-factor graph i.e. T=1, i.e. each three factor graph has e=18 and 2-factor graph has e=12. Hence the theorem is true for m=2

Let us consider that the theorem is true for m=k. Then the form of the complete graph is K_{6k}, in which number of vertices n=6k, number of edges $e=18k^2-3k$ and number of three regular sub-graph S=2k-1. Now, if we put m=k+1, then the complete graph is $K_{6(k+1)}$ where number of vertices n=6(k+1)=6k+6 and number of edges $e=18(k+1)^2-3(k+1)=1=18k^2+33k+15$. Similarly, S=2(k+1)-1=2k+1.

Now, m≥1=>k+1≥1=>k≥0, which shows that S=2k+1 for k≥0.

Hence the theorem is true for all m≥1.

2.1. Algorithm:

INPUT: Let G be a complete weighted graph having n=6m-2/6m+2/6m vertices and edges $e=18m^2-15m+3/18m^2+9m+1/18m^2-3m$. for m≥1.

OUTPUT: To find least cost route.

The following steps are considered

Case-I (For the graph K_{6m-2} for m≥1)

Step1: Study the weighted edges of K_{6m-2} for m≥1.

Step2: If there exists at least one sub-graph of 3-factor having n= 6m-2 vertices then select consecutive minimum weighted edges e=9m-3 out of the edges e=18m^2 -15m+3, and then go to Step3; otherwise go to Step5.

Step3: Study the Hamiltonian circuits from the sub-graph (3-factor) as discussed in Step2.

Step4: The minimum weighted Hamiltonian circuit is found from Step3, studying only the three Hamiltonian circuits as the 3-regular sub-graph (graph) always has 3 Hamiltonian circuits [6].

Step5: Sub graph having 3-factors contains other than the minimum weights, as discussed in Step2, then go to Step6

Step6: Apply the following procedure (i-vii) **[7]**

Step i. Find a regular Hamiltonian sub- graph of degree three of the complete graph K_{2m+2} for m ≥2. [One can find different types of regular sub-graph of degree three]

Step ii. Sum the weights of each row or column associated with the vertices of the regular sub-graph obtained from Step 2. Let them be considered as S_1, S_2, S_3, S_4 ………………S_m.

Step iii. Take the average of the sum of the weights as discussed in step 3 i.e. $\hat{S}$= (S_1+S_2+………. +S_i+……+S_m)/m.

Step iv. If the average($\hat{S}$) as discussed in the step iii is exactly equal to any one sum of the weights of the vertices of row or column, then consider that vertex as a initial vertex i.e. $\hat{S}$=S_i,

Step v. From the initial vertex, go to the 2nd vertex through the minimum weighted path.

Step vi. Continue the same procedure for the selection of vertex (m+1) for all m≥2 other then the initial vertex, and go to the next vertex by minimum weighted path.

Step vii. After selection of (m+1) vertices for all m≥2, we must go to the remaining vertices of the graph to form the Hamiltonian circuit which is the required Hamiltonian circuit (that is least cost route).

Step7: Stop.

Case-II (For the graph K_{6m+2} for m≥1)

Step8: Study the weights of the graph K_{6m+2} for m≥1.

Step9: Find a 1-factor from K_{6m+2} for m≥1, in such a way that the greatest weights must present in the one factor and, also observed that the number of edges in 1-factor are 3m+1 for m≥1.

Step10: Delete the one factor sub-graph from K_{6m+2} for m≥1. Then we will exist a graph whose number of vertices n=6m+2, number of edges $e=18m^2+3m$.

Step11: Then go to Step 2 to Step 7.

Case-III (For the graph K_{6m} for m≥1)

Step12: Study the weights of the graph K_{6m} for m≥1.

Step13: Find a 2-factor from K_{6m} for m≥1, in such a way that the greatest weights must present in the 2 factor and, also observed that the number of edges in 2-factor are 3m for m≥1.

Step14: Delete the 2-factor sub-graph from the graph K_{6m} for m≥1; then there exist a graph whose number of vertices n=6m, number of edges $e=18m^2-6m$.

Step15: Then go to Step2 to Step 7.

2.2. Experimental Results: Let us consider the following examples for non repeated weighted edges.

Example-1: This example is taken for K_{6m-2} when m =2. From the table -1, we have a complete graph K_{10} which is shown in Figure-4.

	A	B	C	D	E	F	G	H	I	J
A	∞	2	18	12	19	20	21	22	23	11
B	2	∞	3	24	13	25	26	27	28	29
C	18	3	∞	4	30	14	31	32	33	34
D	12	24	4	∞	5	35	36	37	38	39
E	19	13	30	5	∞	6	40	41	42	43
F	20	25	14	35	6	∞	7	44	45	46

G	21	26	31	36	40	7	∞	8	15	47
H	22	27	32	37	41	44	8	∞	9	16
I	23	28	33	38	42	45	15	9	∞	10
J	11	29	34	39	43	46	16	47	10	∞

Table-1

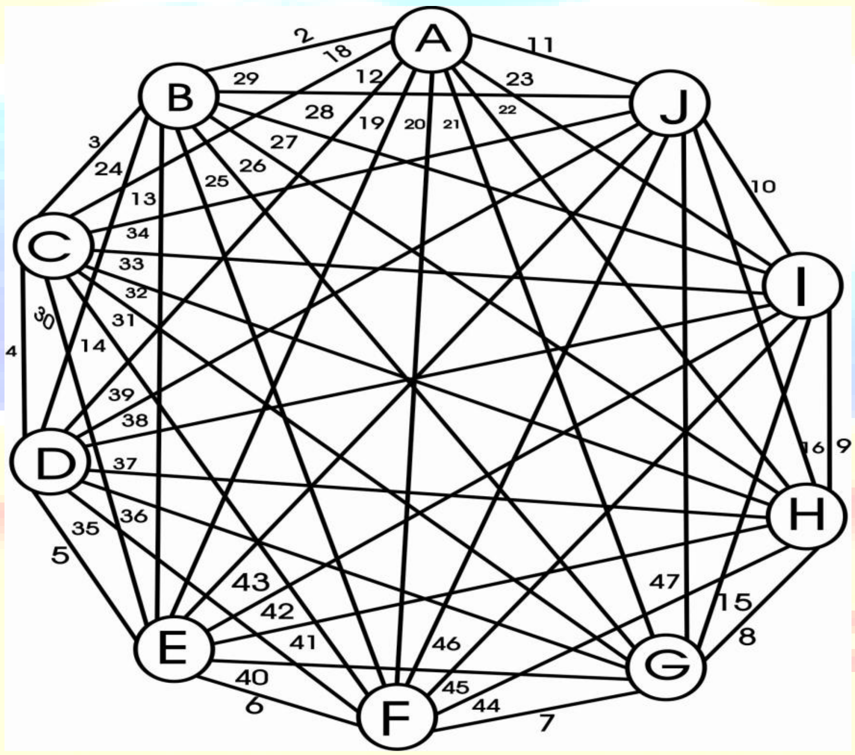

Figure-4

Now, applying the Step-2 of the algorithm, we obtain the Table-2 which coresponds the graph of figure-5.

	A	B	C	D	E	F	G	H	I	J
A	∞	2		12						11
B	2	∞	3		13					
C		3	∞	4		14				
D	12		4	∞	5					
E		13		5	∞	6				
F			14		6	∞	7			
G						7	∞	8	15	
H							8	∞	9	16
I							15	9	∞	10
J	11							16	10	∞

Table-2

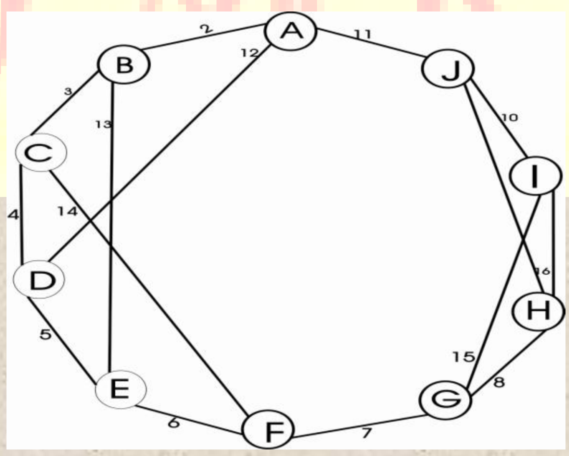

Figure-5

Hence from the three regular sub graph as shown in Figure-5, we have the minimum weighted Hamiltonian circuits out of these three Hamiltonian circuits as

$A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow G \rightarrow H \rightarrow I \rightarrow J \rightarrow A$

with weight equal to 65.

Example-2: This example is taken for K_{6m+2} when m =2.

	A	B	C	D	E	F	G	H
A	∞	30	19	40	12	21	11	2
B	30	∞	8	35	9	22	25	60
C	19	8	∞	3	23	51	24	61
D	40	35	3	∞	4	70	10	50
E	12	9	23	4	∞	7	15	20
F	21	22	51	70	7	∞	5	13
G	11	25	24	10	15	5	∞	14
H	2	60	61	50	20	13	14	∞

Table-3

From Table-3, we can construct a one factor sub graph in such a way that the maximum weights present in this 1-factor and accordingly we have DF=70, CH=61. Thereafter, we must select AB=30 and EG=15 for completing the one factor sub graph. Now, we delete this one factor sub graph from K_8 and then proceeding the remaining steps as discussed in Example1.Then we have the minimum weighted Hamiltonian circuit as

$A \rightarrow H \rightarrow F \rightarrow G \rightarrow D \rightarrow C \rightarrow B \rightarrow E \rightarrow A$

with weight equal to 62.

Example -3: This example is taken for K_{6m} when m=1

	A	B	C	D	E	F
A	∞	10	19	7	40	9
B	10	∞	12	28	19	33
C	19	12	∞	26	20	22
D	7	28	26	∞	25	30
E	40	19	20	25	∞	15
F	9	33	22	30	15	∞

Table-4

From Table-4, we can construct a two factor sub graph in such a way that the maximum weights are considered in descending order, which are AE=40, BF=33, DF=30, BD=28, CD=26, then we must select AC=19 for completing the two factor sub graph. Now, we delete this two factor sub graph from K_6 and then using the remaining steps proceeded in Example-1.Then we have the minimum weighted Hamiltonian circuit as

$$A \rightarrow D \rightarrow B \rightarrow C \rightarrow E \rightarrow F \rightarrow A$$

with weight equal to 81.

2.3: Conclusion: We first discuss the factors of the graphs and apply them in the algorithm for the solution of TSP. More results will be discussed in near future.

References:

[1] Kalita, B. : "Some Investigation on Graph Theory, PhD thesis". Finance India. Vol XIX, NO-4DEC (P.P. 1430-1438), 2005

[2] Kalita, B. : "Sub Graph of Complete Graph" Proceeding, International Conference on Foundation of computer Science. Lasvegas, U.S.A. p.p. 71-77, 2006

[3] Kalita, B. : "Application of three edges extension of planar (TEEP) graph for the Solution of traveling salesman problem" Proc. 48th ISTAM conference 2003.

[4] Kalita, B. : "SCN with DCS tree with cover and Hamiltonian circuit" Proc international workshop on Telemetric 1 - 5th June,1995 p.p. 163 - 168, 1995.

[5] Deo, N. & Hakim, S.L.: "The shortest generalized Hamiltonian tree" Third Annual Alterton Conference, University of Illinois. (p.p. 879-888), 1965

[6] Dutta. Anupam, Kalita. B., Baruah. H.K., : "Regular sub-graph of complete graph" International Journal of Applied Engineering Research. Vol.5. No-8, (2010) p.p. 1315-1323.

[7] Dutta. A, Kalita.B, Baruah.H.K, : "Regular Planar Sub-Graphs of Complete Graph and Their Application" .IJAER, Vol-5 no-3, (2010), p.p. 377-386.

[8] Dutta. A, Kalita.B, Baruah.H.K, : "Crossing and Thickness of Special type of Non-Planar Graph". Journal of Pure Mathematics, University of Calcutta, Vol-24,(2007), p.p. 39-47.

[9] Choudhury. J.K., Kalita, B, : "An Algorithm for TSP" Bulletin of Pure and Applied Sciences, Vol 30 E (Math &Stat.) Issue (No.1) 2011: P. 111-118

[10] Harary, F. Graph Theory" Narosa Publishing House, 1995.